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# ESTIMATING THE SOLIDIFICATION/MELTING TIMES OF CYLINDRICALLY SYMMETRIC REGIONS

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**Abstract**—A simple, explicit algorithm previously developed to produce accurate solutions of phase change problems in one space dimension, is extended to cope with circular regions with spatially uniform boundary conditions. On study of numerically predicted results a single non-dimensional expression, which provides a prediction of the solidification/melting time of a circular cylinder, is derived. This expression is subsequently used to provide upper and lower bounds on solidification/melting times for general symmetric cylindrically shaped regions with spatially uniform boundary conditions.

## NOMENCLATURE

- $c$, specific heat;
- $E$, percentage error spread (see Section 4);
- $dm$, diameter of minor axis;
- $H$, enthalpy;
- $L$, latent heat;
- $R(t)$, radial position of phase change boundary;
- $r$, radial coordinate;
- $r_a$, radius of approximating cylinder (see Section 4);
- $r_l$, radius of lower bounding cylinder (see Section 4);
- $r_u$, radius of upper bounding cylinder (see Section 4);
- $T$, temperature;
- $Tm$, phase change temperature;
- $Tw$, fixed surface temperature;
- $T_0$, initial temperature;
- $t$, time;
- $t_s^*$, non-dimensional phase change time;
- $\rho$, density;
- $(\#)^*$, non-dimensional variable.

## 1. INTRODUCTION

HEAT conduction problems involving a change of phase are usually solved by numerical methods. The major difficulty to overcome in generating a numerical solution is in the representation of the discontinuity of the temperature gradient at the phase change boundary. A popular way to avoid this difficulty involves the use of the enthalpy method [1–8]. Here the governing equations are reformulated in terms of the enthalpy, $H$ (i.e. the sum of the sensible and latent heats). This removes the need to directly trace the position of the moving boundary and, hence, eliminates the numerical problems associated with the discontinuity of the temperature gradient.

Although two recent comprehensive reviews [9, 10] have cited enthalpy methods as the best approach for a wide variety of problems, numerically induced oscillations have been observed [2, 11–13] in the predictions of the standard enthalpy methods. Recently, we [13, 14] have proposed a simple, explicit technique for one-dimensional problems, based upon the enthalpy method, which both eliminates the numerically induced oscillations and produces accurate predictions. Unfortunately, except in the cases of rectangular and circular regions, analysis of phase change problems in two-dimensional regions is not only complicated, but also there is no obvious method to assess the reliability of the predictions. The aim of this paper is to describe an extension to the authors one-dimensional method which provides a means to estimate upper and lower bounds to melting/solidification times for two-dimensional regions with cylindrical symmetry and spatially uniform boundary conditions.

## 2. THE GOVERNING EQUATIONS AND NUMERICAL ALGORITHM

Phase change problems in cylindrical two dimensional regions with radial symmetry may be formulated in terms of enthalpy as follows:

$$\rho \frac{\partial H}{\partial t} = K\left\{\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right\} \tag{1}$$

where $\rho$ is the density, $K$ the thermal conductivity and $T$ the temperature. The enthalpy is related to the temperature by

$$T = \begin{cases} H/c & H \le cTm \\ Tm & cTm < H < cTm + L \\ (H-L)/c & H \le cTm + L \end{cases} \tag{2}$$

where $c$ is the specific heat, $L$ is the latent heat of the phase change and $Tm$ is the phase change temperature.

Using central differences, the explicit form of the approximation to equation (1) is given by

$$H_i^{j+1} = H_i^j + P[(1+1/2i)\,T_{i+1}^j - 2T_i^j + (1-1/2i)\,T_{i-1}^j], \ (i = 1, \ldots, n) \tag{3}$$

where $P = K\Delta t/\rho(\Delta r)^2$, $\Delta t$ is the time step, $\Delta r$ is the radial distance step, and $T_i^j$, $H_i^j$ are the temperature and enthalpy, respectively, at the position $r = i\Delta r$ and

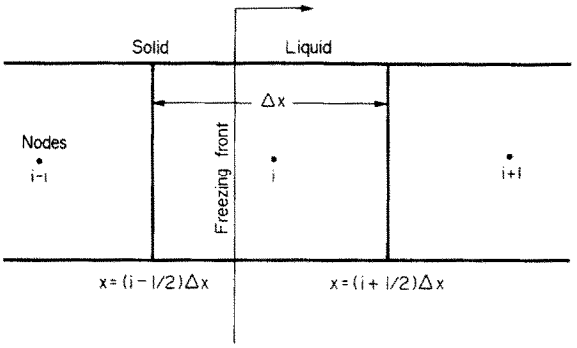


FIG. 1. The movement of a freezing (i.e. phase change) front through a one-dimensional discretized region.

time $t = j\Delta t$. At the point $r = 0$ (i.e. when $i = 0$) the above scheme becomes

$$H_0^{j+1} = H_0^j + 4P(T_1^j - T_0^j) \tag{4}$$

since [15]

$$\lim_{r\to 0} \frac{1}{r}\frac{\partial T}{\partial r} = \left.\frac{\partial^2 T}{\partial r^2}\right|_{r=0}$$

The basis of the one-dimensional linear algorithm for phase change problems that we have proposed [13, 14] is to relate the numerical value of the enthalpy at a node to the position of the phase change boundary. For the majority of problems, unless the time step is large or the size step very small, as the boundary passes through a node its enthalpy value will remain in the range $[cTm, cTm + L]$ for a number of time steps. This time period may be identified as the time taken for the element of thickness $\Delta x$ about the node $i$ to undergo its phase change [i.e. for the boundary to move from $(i - 1/2)\,\Delta x$ to $(i + 1/2)\,\Delta x$, as shown in Fig. 1]. Starting from this observation, the phase change boundary may be positioned on the node $i$ whenever the nodal enthalpy, $H_i = cTm + L/2$. The idea of relating the phase change boundary position to the above nodal enthalpy value led to the development of a simple, yet accurate procedure to trace the boundary through a one-dimensional region.

A similar procedure may be developed to trace the radial movement of a phase change front, $R(t)$, of a circular region with spatially uniform boundary conditions (i.e. radially symmetric). The time that the moving boundary reaches node $i$ is evaluated as follows. Suppose (in a freezing problem) the nodal enthalpy as predicted by the numerical solution of equations (2)–(4) is such that $H_i^{j+1} < cTm + L/2$ and $H_i^j > cTm + L/2$. Then the radial position of the phase change boundary will be $R(t_i) = i\Delta r$ where

$$t_i = (j + x)\,\Delta t \tag{5}$$

and $x$, which lies between 0 and 1, is evaluated by a linear interpolation in time, i.e.

$$x = \frac{(L/2 + cTm - H_i^j)}{(H_i^{j+1} - H_i^j)}. \tag{6}$$

Although this algorithm has been shown to work well for truly one-dimensional problems it should be established that it also works for cylindrical problems. To fulfill this objective a simple test problem is proposed where at time $t = 0$ the region $r > a$ is in the liquid phase at its melting temperature $Tm > 0$. For $t > 0$ the surface $r = a$ is maintained at zero temperature so that the position, $R(t)$, of the interface between the liquid and solid phases satisfies $R(t) \geq a$. If the thermal properties of the solid and liquid phases are taken as equal and the ratio $L/c$ is large, then an approximate solution for $R(t)$ is given by [16]

$$2R(t)^2 \ln [R(t)/a] - R(t)^2 + a^2 = \frac{4K\,Tm\,t}{L\rho}. \tag{7}$$

This solution is illustrated by the continuous line in Fig. 2 using the following thermal and physical properties $a = 0.5$ m, $Tm = 10°C$, $K = 2$W/m K, $c = 2.5$ MJ/kg K, $\rho = 1$ kg/m$^3$, $L = 100$ MJ/kg. The corresponding numerical solution which was evaluated using $\Delta t = 1$ h and $\Delta r = 0.125$ m, is shown as the short dashed line in Fig. 2 and is close to the approximate analytic solution. The comparison is further improved for the numerical solution generated for the case $L/c = 100$ (i.e. the dot-dash line in Fig. 2). As such it would appear that the method proposed above provides accurate solutions to circular cylindrically shaped regions with spatially uniform boundary conditions. At this point it should be noted that the interpolation given by equation (6) is linear, i.e. no radial effects are included. The results of the above test problem, however, indicate that this assumption does not greatly affect the accuracy of the proposed method.

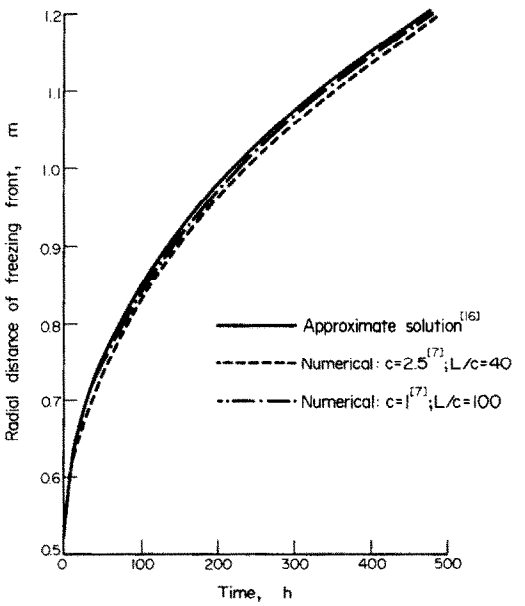


FIG. 2. Comparison of approximate analytic and numeric solutions for freezing around a cylinder.

Table 1. Comparison of dimensionless solidification times as predicted by the numerical method and equation (12)

| $L^*$ | $T_0^*$ | Dimensionless solidification time | | Difference (%) |
|---|---|---|---|---|
| | | Numerical | Equation (12) $t_s^*$ | |
| 2 | 0.2 | 0.663267 | 0.660 | 0.495 |
| 2 | 1 | 0.731207 | 0.724 | 0.995 |
| 2 | 2 | 0.794881 | 0.804 | −1.134 |
| 20 | 0.2 | 5.18902 | 5.1870 | 0.039 |
| 20 | 1 | 5.22983 | 5.215 | 0.284 |
| 20 | 2 | 5.26785 | 5.25 | 0.340 |
| 50 | 0.2 | 12.7078 | 12.732 | −0.19 |
| 50 | 1 | 12.7854 | 12.70 | 0.672 |
| 50 | 2 | 12.8623 | 12.66 | 1.598 |

## 3. THE PHASE CHANGE TIME OF A CIRCULAR CYLINDER

For many practical problems involving a change of phase, the prime interest is to evaluate the solidification or melting time of the region. Consider the following solidification problem, at time $t < 0$ an infinitely long cylinder with a circular cross-section is at a uniform temperature $T_0 > Tm$ the phase change temperature of the cylinder. At time $t = 0$ the surface of the cylinder is lowered to and fixed at a temperature $Tw < Tm$ so that solidification begins. This problem can be described by equations (1) and (2) which can be written in the following non-dimensional form

$$\frac{\partial H^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial r^*} + \frac{1}{r^*}\frac{\partial T}{\partial r^*} \quad 0 < r^* \leq 1 \tag{8}$$

where

$$T^* \begin{cases} H^* & H^* \leq 0 \\ 0 & 0 < H^* < L^* \\ H^* - L^* & H^* \geq L^* \end{cases} \tag{9}$$

with the boundary condition

$$T^*(0, t^*) = -1 \tag{10a}$$

and the initial condition

$$T^*(r^*, 0) = T_0^*. \tag{10b}$$

The dimensionless variables are defined by

$$t^* = Kt/R^2 \rho c; \quad r^* = r/R$$

$$L^* = \frac{L}{c(Tm - Tw)}; \quad H^* = \frac{H}{c(Tm - Tw)} \tag{11}$$

$$T^* = (T - Tm)/(Tm - Tw)$$

$$T_0^* = (T_0^* - Tm)/(Tm - Tw)$$

where $R$ is the radius of the circular cross-section.

The numerical scheme outlined and verified above has been used to solve solidification problems of the type described by equations (8)–(10). The numerical results from this scheme indicate that the dimensionless solidification time $t_s^*$ for the circular cylinder may be approximated by the following expression, viz.

$$t_s^* = (0.14 + 0.085\, T_0^*) + (0.252 - 0.0025\, T_0^*)\, L^*. \tag{12}$$

This approximation predicts solidification times within 1% of the numerical predictions, c.f. Table 1, when

$$0 \leq T_0^* \leq 2$$

and

$$2 \leq L^* \leq 50.$$

Hence for a wide range of cases the solidification (or melting) time of a cylindrical region with circular cross-section and spatially uniform boundary conditions may be readily found on use of equations (11) and (12).

## 4. APPLICATION TO GENERAL SYMMETRICALLY SHAPED CYLINDERS

In a case where the cross-section of the cylinder is non-circular calculation of the solidification/melting time may be difficult. As a first step prediction, however, use of equation (12) may provide a reasonable estimate of this time.

Important factors in deciding the phase change time of a symmetric cylindrical region are

(a) its cross-sectional area, $A$;
(b) the ratio of the circumference to the cross-sectional area $C/A$; and
(c) the shape of the cross-section.

When the cylinder's cross-section is long and thin, e.g. a long, thin ellipse or rectangle, a very reasonable estimate for the phase change time can be determined directly from a one-dimensional solution. As the cross-sectional shape of the cylinder approaches a circle, however, this one-dimensional approach will predict a phase change time which is greater than the true phase change time. On defining the minor axis, $dm$, of a cross-section of a symmetric cylinder to be the shortest possible straight line which passes through the cross-section's center of gravity, a circle with radius $r = dm/2$ will be completely contained within the cross section. The phase change time of a circular cylinder with cross-sectional radius $dm/2$ will therefore be shorter

than the phase change time of the original cylinder. On the other hand a circular cylinder with the same cross-sectional area as the original cylinder will take longer to undergo its phase change. This is because the ratio of circumference to area is a minimum for a circle. So a circular cylinder of diameter $dm$ which fits inside the original cylinder solidifies or melts faster and a circular cylinder with the same cross-sectional area as the original cylinder takes longer to solidify or melt.

Consider the following solidification problem. A long square shaped channel with cross-sectional area 1 m$^2$, contains liquid initially at 2°C. The walls of the channel are held fixed at $-10$°C for times, $t > 0$, so that the liquid in the channel slowly solidifies. The phase change temperature is 0°C and the thermal properties of the material in the channel are as in section 2 above. A lower bound on the solidification time of the channel can be found on calculating the solidification time of a circular cylinder of radius 0.5 m, with the same boundary conditions and thermal properties as the channel, by equations (11) and (12). An upper bound on the channel's solidification time is found on calculating the solidification time by equations (11) and (12) of a circular cylinder with cross-sectional area 1 m$^2$, i.e. a radius of 0.5642 m, and identical conditions to the channel. The upper and lower bounds on the solidification time of the square channel by use of the above approach are 128.5 h and 101 h, respectively.

A direct numerical solution for the freezing of the square channel was generated previously by the authors [13, 14] using a modification to the original algorithm. This solution predicts a solidification time of 119.4 h which, as expected, lies between the upper and lower bounds calculated above.

Although the upper and lower bounds provide an indication of the solidification time, they are only accurate to within $\sim 15\%$. A reasonable estimate for the solidification time may be evaluated by using an approximating circular cylinder whose radius is given by $r_a = (0.5 + 0.5642)/2 = 0.5321$ m. The predicted solidification time in this case is 114.3 h, which is within 5% of the time predicted by the direct solution of the two dimensional region. All the above results are summarized in Table 2.

Use of the above method to calculate the solidification/melting time of a general symmetrically shaped cylinder in a large number of cases may only provide a first order indication of the phase change time. The relative accuracy of the upper and lower bounds depends on the difference in the radii of the approximating cylinders. More precisely the 'percentage error spread', i.e. the sum of the percentage errors in lower and upper bounds, may be defined as

$$E = \frac{r_u^2 - r_l^2}{r_a^2} \times 100$$

where $r_u$ and $r_l$ are the radii of the upper and lower approximating cylinders and

$$r_a = (r_u + r_l)/2.$$

The value of $E$ indicates the accuracy of using approximating cylinders for upper and lower bounds on the phase change time of a cylinder with non-circular cross-section. For the example of the square channel outlined above $E = 24.1$. When the cross-section is a pentagon with sides of 1 m $E = 14.5$, which indicates, in this case, that the method of 'approximating cylinders' will be more accurate than for the square.

## 5. CONCLUSIONS

A modification to the enthalpy method proposed by the authors elsewhere [13, 14] has been extended to cope with cylindrical problems containing a circular cross-section and spatially uniform boundary conditions. After the study of numerically predicted results a single non-dimensional expression was derived which estimated the numerically predicted phase change time of a circular cylindrical region to an accuracy of 1%. This expression was then used to provide upper and lower bounds for the solidification times of symmetrically shaped cylinders with spatially uniform boundary conditions. The procedure was tested on a cylinder with a square cross-section for which a direct numerical solution was available. In this

Table 2. Comparison of solidification times

| Solution type | Solidification time (h) | Difference from direct solution (%) |
|---|---|---|
| Direct numeric solution of square channel, area = 1 m$^2$ | 119.4 | 0 |
| Lower bound approximating circular cylinder, $r_l = 0.5$ | 101 | 15.4 |
| Upper bound approximating circular cylinder, $r_u = 0.5642$ | 128.5 | 7.6 |
| Averaged approximating circular cylinder, $r_a = (r_u + r_l)/2 = 0.5321$ | 114.3 | 4.5 |

case, the upper and lower bounds were within 15% of the solidification time predicted by the direct scheme. However, a simple refinement produced an estimate of the solidification which was within 5% of the solidification time predicted by the direct numerical scheme.

For the square channel considered in this paper the procedure to determine upper and lower bounds on the solidification time proves crude. This is because the actual solidification time can be estimated with some confidence. The method outlined in this paper, however, is not intended to be a rigorous approach to deriving a phase change time of a symmetric cylinder. It is intended more to be a quick, pragmatic means of estimating the phase change time of a cylindrical region where other methods of analysis cannot be readily used. For problems where the cross-section of the cylinder is not a simple shape, then, depending on the accuracy required, the procedure outlined in this paper could prove to be a useful tool in estimating the nature of freezing and melting in symmetrically shaped cylindrical regions with spatially uniform boundary conditions.

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## ESTIMATION DES TEMPS DE SOLIDIFICATION ET DE FUSION DE DOMAINES A SYMETRIE CYLINDRIQUE


**Résumé** — Un algorithme explicite et simple précédemment développé pour fournir des solutions précises des problèmes de changement de phase dans un espace monodimensionnel est étendu pour couvrir les domaines circulaires avec des conditions aux limites spatialement uniformes. On obtient une expression unique adimensionnelle qui fournit l'estimation du temps de solidification ou de fusion d'un cylindre circulaire. Cette expression est utilisée pour avoir les limites supérieure et inférieure des temps de solidification/fusion pour des domaines à symétrie cylindrique avec des conditions aux limites spatialement uniformes.


## ABSCHÄTZUNG DER ERSTARRUNGS- BZW. SCHMELZZEITEN VON ROTATIONSSYMMETRISCHEN GEBIETEN


**Zusammenfassung**—Ein vor kurzem entwickelter, expliziter Algorithmus, der genaue Lösungen von eindimensionalen Phasenänderungsproblemen liefert, wurde für rotationssymmetrische Gebiete mit räumlich gleichförmigen Randbedingungen erweitert. Aufgrund einer Analogie numerisch berechneter Werte wurde ein einfacher dimensionsloser Ausdruck entwickelt, der die Berechnung der Erstarrungs-/Schmelzzeiten eines Kreiszylinders zuläßt. Dieser Austruck wird später dazu benutzt, die Unter- und Obergrenzen von Erstarrungs- bzw. Schmelzzeiten für allgemeine rotationssymmetrische Gebiete mit räumlich gleichförmigen Randbedingungen anzugeben.

## РАСЧЕТ ВРЕМЕНИ ЗАТВЕРДЕВАНИЯ И ПЛАВЛЕНИЯ ЦИЛИНДРИЧЕСКИХ СИММЕТРИЧНЫХ ОБЛАСТЕЙ

**Аннотация** — Предложенный ранее простой алгоритм в явной форме для точного решения задач с фазовыми изменениями в одном пространственном измерении применен к случаю круговых областей с пространственно однородными граничными условиями. В результате численного исследования выведено безразмерное выражение, которое дает возможность рассчитывать время затвердевания или плавления круглого цилиндра. Затем с помощью данного выражения определены верхняя и нижняя границы времени затвердевания и плавления симметричных областей цилиндрической конфигурации с указанными выше граничными условиями.